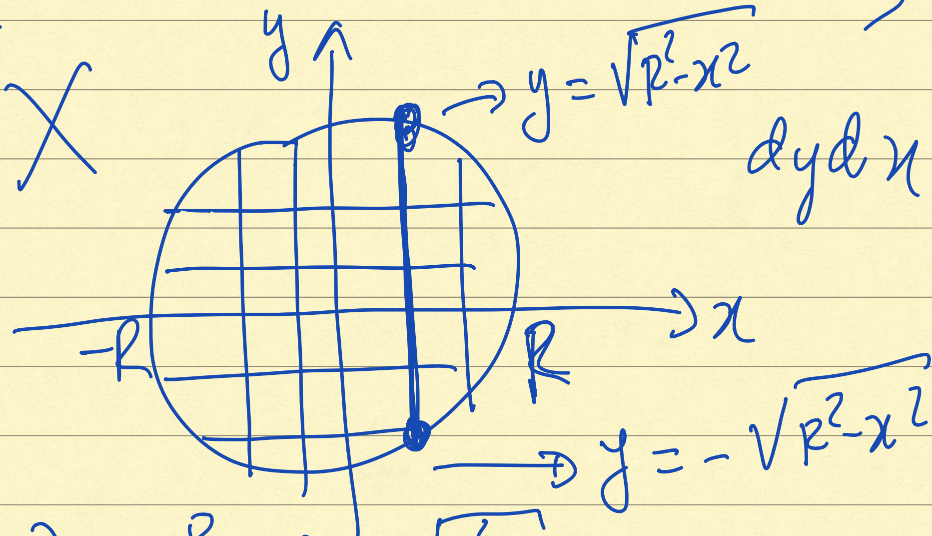
Mudança de variáveis de integração.

Coordenadas polares em $\mathbb{R}^2$

$$X = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\}$$



$dy\,dx$

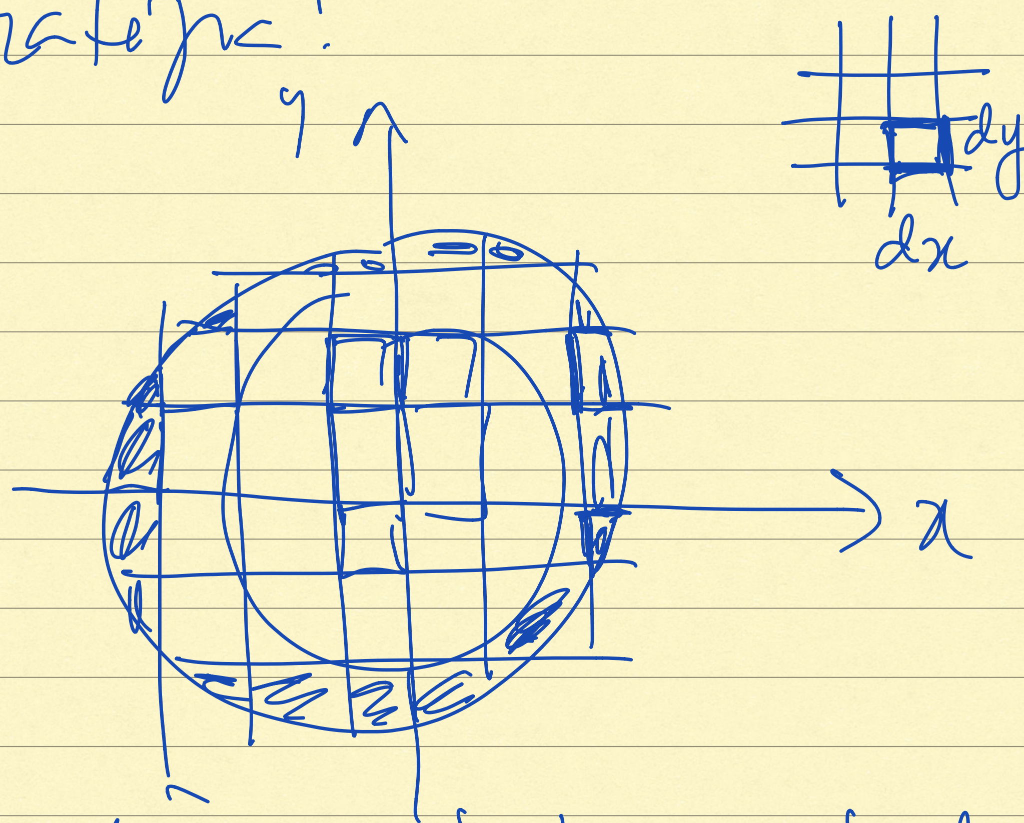
$$\mathrm{vol}_2(X) = \int_{-R}^{R} \left(\int_{\underbrace{-\sqrt{R^2-x^2}}}^{\sqrt{R^2-x^2}} dy \right) dx$$

$$= \int_{-R}^{R} 2\sqrt{R^2 - x^2}\, dx = 4 \int_0^R \sqrt{R^2 - x^2}\, dx$$

$$4\int_0^R \sqrt{R^2-x^2}\,dx \quad ? \quad \text{CDI-I}$$

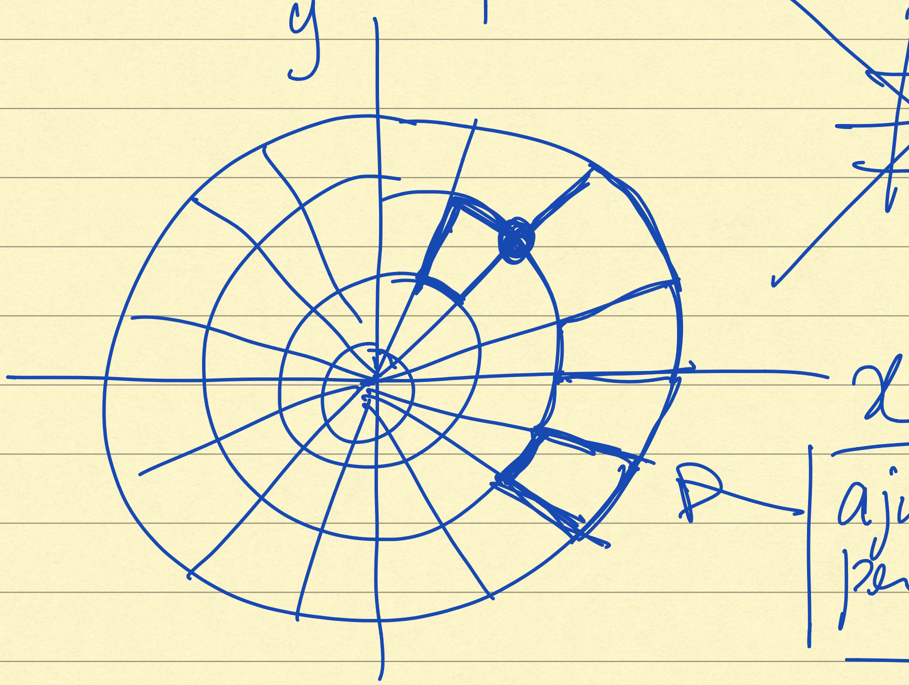
→ substituição ou mudança de variável

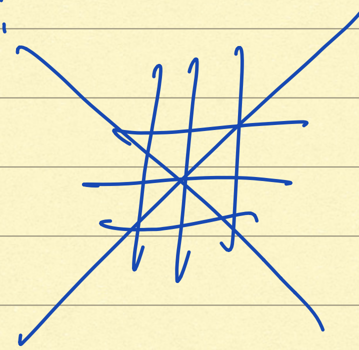
Fazer uma mudança de estratégia!



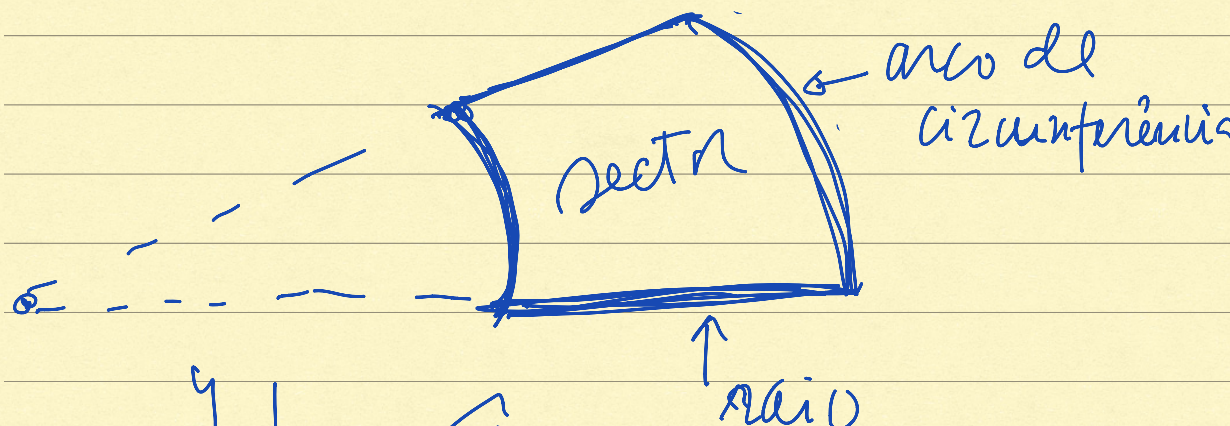
" $\square$ " não se ajusta ao círculo

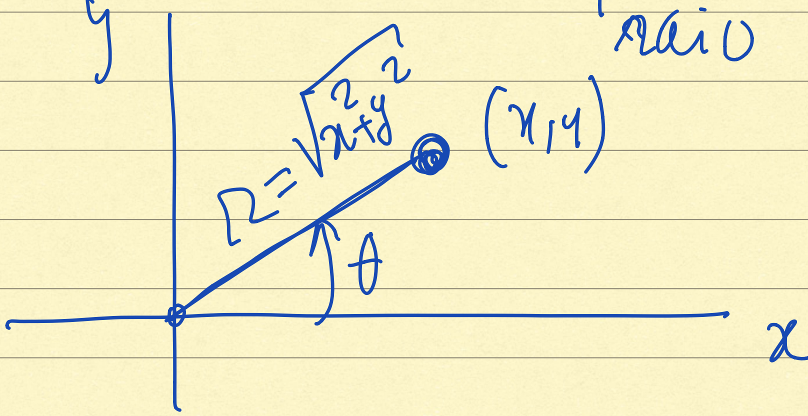
Ver de outra forma:



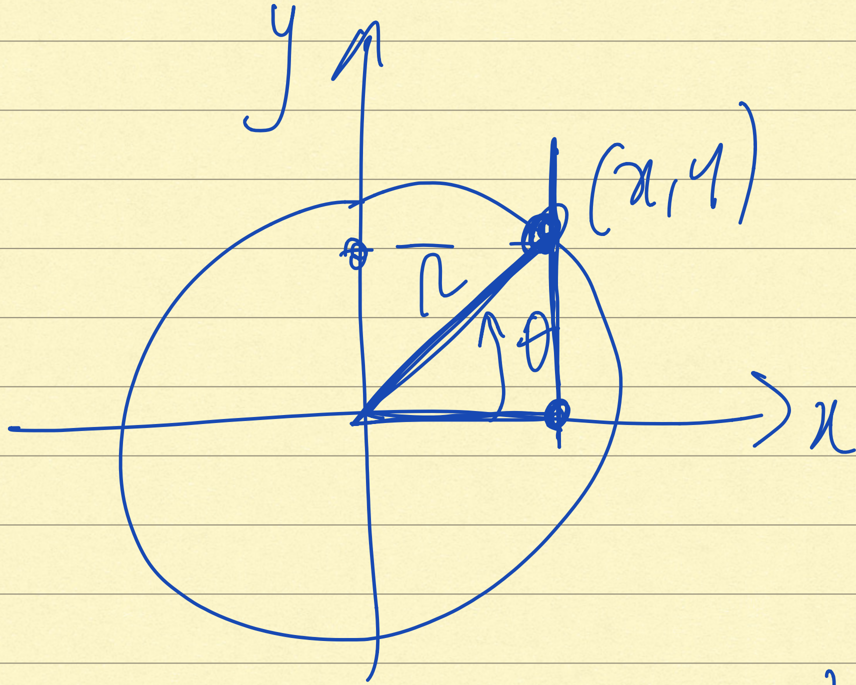


ajuste perfeito





"θ" téta



r = distância de (x,y) à origem.

$$r = \sqrt{x^2+y^2}$$

$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases} \qquad \boxed{(x,y) = (r\cos\theta, r\,\mathrm{sen}\,\theta)}$$

$\iint dx\,dy$ ~ ?? $\iint dr\,d\theta$

X ?

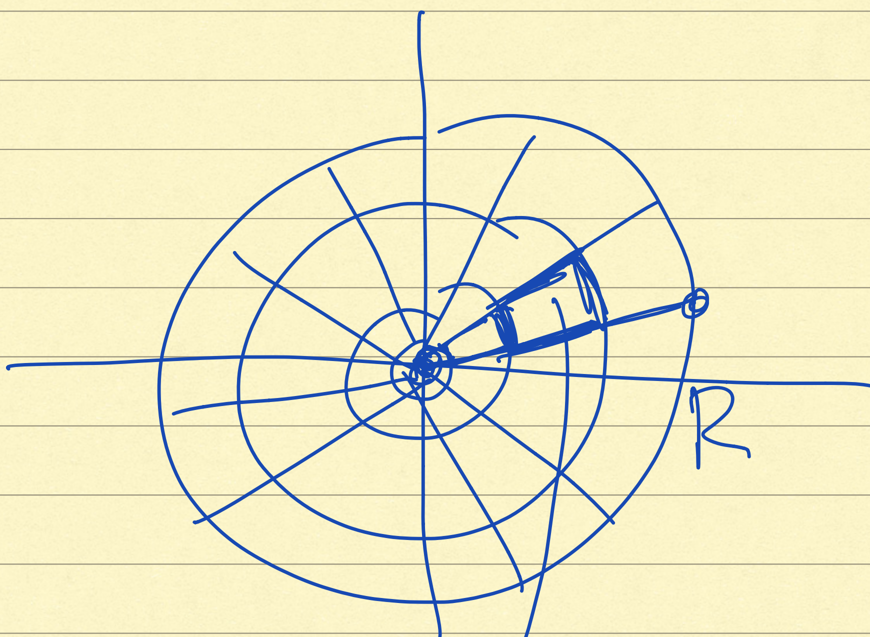
CDI-I:

$$\underbrace{\int_a^b f(x)\,dx}_{\text{difícil}} = \int_{\dots}^{\dots} f(g(t))\,|g'(t)|\,dt$$

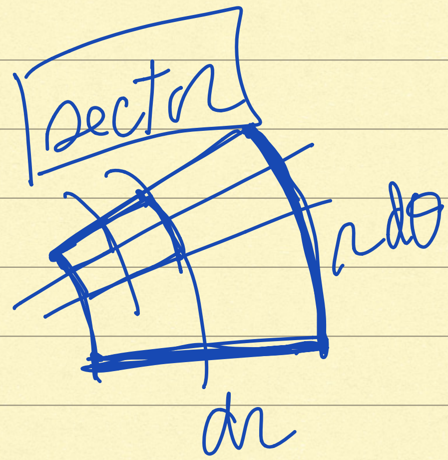
fácil

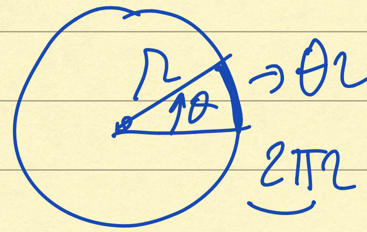
"preço"

$$\boxed{x = g(t)}$$



"parece um rectângulo"





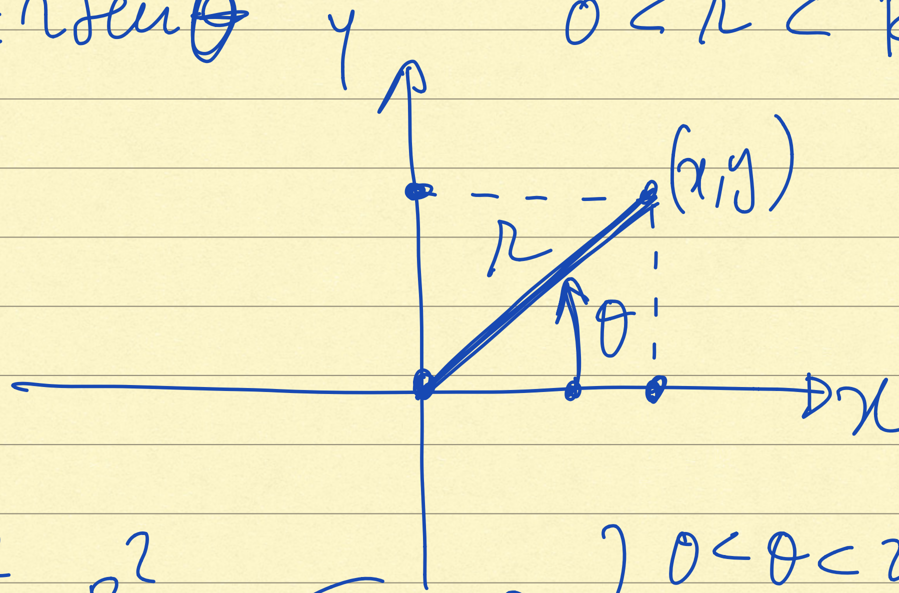
area for sector:

$$\boxed{r\,dr\,d\theta}$$

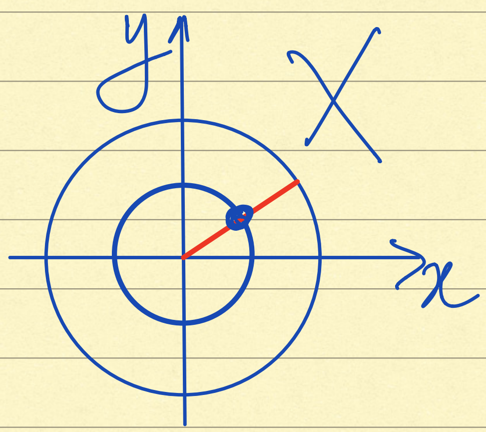
$$dxdy \sim r\,dr\,d\theta$$

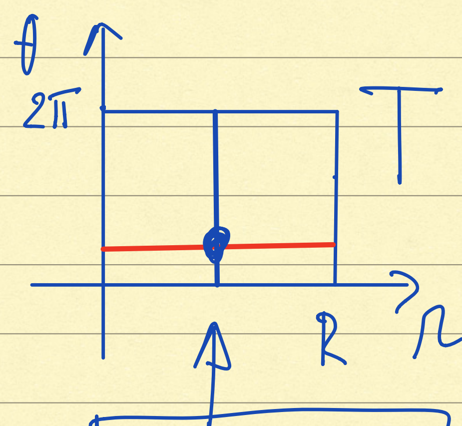
$$\iint dxdy \overset{=}{\longleftrightarrow} \iint r\,dr\,d\theta$$

$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases} \qquad 0 < \theta < 2\pi \qquad 0 < r < R$$



$$x^2 + y^2 < R^2 \quad \longleftrightarrow \quad \begin{cases} 0 < \theta < 2\pi \\ 0 < r < R \end{cases}$$





$\boxed{\text{círculo}}$

"horrível" para o integral!

$\boxed{\text{intervalo}}$

"bom" para o integral!

$$\iint_X dx\,dy \overset{???}{=} \iint_T r\,dr\,d\theta$$

$$= \int_0^R \left(\int_0^{2\pi} r\,d\theta \right) dr$$

or

$$= \int_0^{2\pi} \left(\int_0^R r\,dr \right) d\theta$$

$$= \int_0^{2\pi} \frac{R^2}{2}\,d\theta = \frac{R^2}{2}\,2\pi$$

$$= \boxed{\pi R^2} \quad //$$

$$(x,y) = (r\cos\theta,\ r\,\text{sen}\,\theta)$$

$$\underset{\text{"velhas"}}{(x,y)} = g(\underbrace{r,\theta}_{\text{"novas"}})$$

r

$\mathbb{R}$

$x = g(t)$

velha

nova

$$g : T \longrightarrow \mathbb{R}^2,\ C^1$$

$$Dg(r,\theta) = \begin{bmatrix} \cos\theta & -r\,\text{sen}\,\theta \\ \text{sen}\,\theta & r\cos\theta \end{bmatrix}$$

$$\det Dg(r,\theta) = r\cos^2\theta + r\,\text{sen}^2\theta$$

$$= r \neq 0$$

$$\iint_X dx\,dy = \iint_T |\det Dg(r,\theta)|\,dr\,d\theta$$

difícil — $\boxed{X \text{ círculo}}$

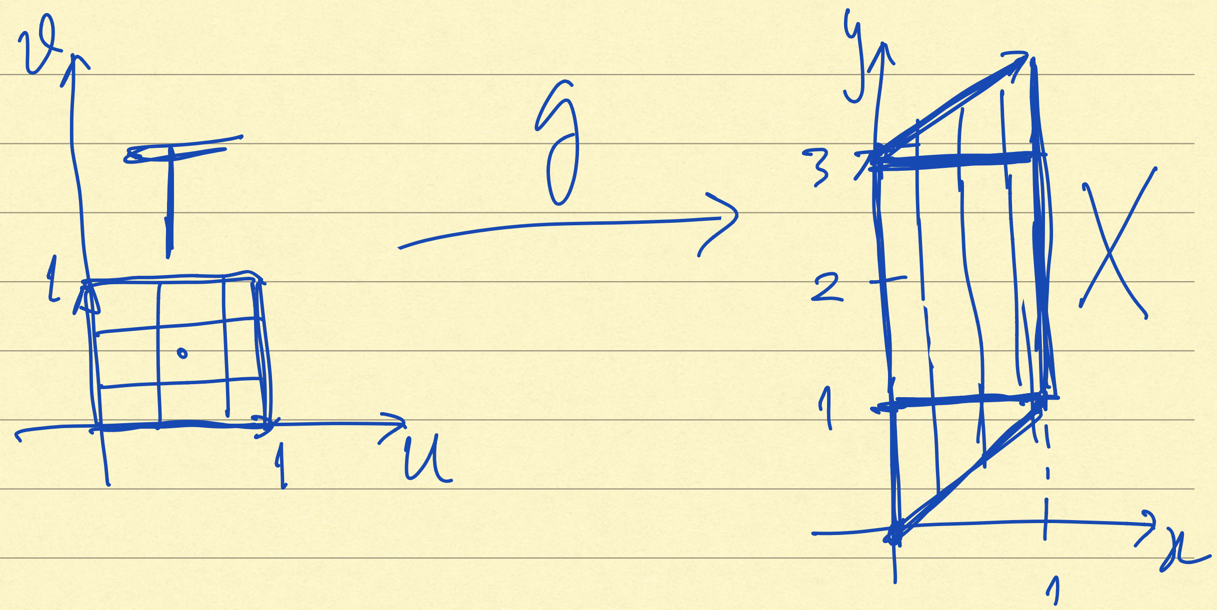
fácil. — $\boxed{T \text{ intervalo}}$

Caso linear : Exemplo

$$g(u,v) = (u, u+3v)$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$(x,y) = g(u,v)$$



$$\begin{cases} x = u \\ y = u + 3v \end{cases}$$

$g(1,0) = (1,1)$

$g(0,1) = (0,3)$

$$X = g(T)$$

$\mathrm{vol}_2(T) = 1$

$\mathrm{vol}_2(X) = 3$

$$\mathrm{vol}_2(X) = 3\,\mathrm{vol}_2(T)$$

$$\det Dg(u,v) = \det \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} = 3 \;//$$

$$\iint_X dx\,dy = \iint_T 3\,du\,dv = \int_0^1 \left(\int_0^1 3\,du \right) dv = 3 \;//$$

$$\boxed{\text{Mudança de variáveis}}:$$

$$(x,y) = g(u,v)$$

$$\iint_X f(x,y)\,dx\,dy = \iint_T f(g(u,v))\,|\det Dg(u,v)|\,du\,dv$$

$$X = g(T)$$

Coordenadas polares (r,θ):

$$(x,y) = (r\cos\theta, r\,\mathrm{sen}\,\theta)$$

$$(x,y) = g(r,\theta)$$

$$\det Dg(r,\theta) = r$$

$$\iint_X f(x,y)\,dx\,dy = \iint_T f(g(r,\theta))\,r\,dr\,d\theta$$

$X: x^2+y^2 < R^2$ difícil!

$T: \begin{cases} 0 < r < R \\ 0 < \theta < 2\pi \end{cases}$ $\boxed{\text{fácil!}}$